

Indian Statistical Institute
First Semester, M.Math II Year
Mid Semestral Examination, 2004-2005
Representation of Groups

Time: 3 hrs

Date:24-09-04

Section I

Answer any three, each question carries 5 marks

1. Show that any T_1 topological group is T_2 . Are all topological groups T_1 or equivalently T_2 ? Justify your answer.
2. Show that $GL_n(\mathbb{C})$ is connected.
3. Let G be a locally compact T_2 group and m be a left Haar measure on G . If A is a compact set such that $m(A) > 0$, then AA and $A^{-1}A$ contain non-empty open sets.
4. Let G be a finite group and $\rho : G \rightarrow GL(V)$ be a representation of G in a vector space V over a field \mathcal{K} . Suppose V is of finite-dimension and $|G| \neq 0$ in \mathcal{K} . Then ρ is completely reducible.
5. Let $V = \mathbb{C}^n$ and $G = S_n$. Let $T : G \rightarrow GL(V)$ be the representation defined by $T(\sigma)(e_1, \dots, e_n) = (e_{\sigma(1)}, \dots, e_{\sigma(n)})$. Show that T is not irreducible and T restricted to $V_0 = \{(x_1, \dots, x_n) \in \mathbb{C}^n / \sum x_i = 0\}$ is an irreducible representation.

Section II

Answer any three, each question carries 10 marks

1. a) Let G be a topological group. If B is a compact set and A is a closed set, prove that AB is a closed set. Is AB closed whenever A & B are closed? If so, prove or if not, give a counter example.
b) Let G be a topological group and H be a subgroup of G . Then
(i) H is T_2 & G/H is $T_2 \Rightarrow G$ is T_2
(ii) H is connected G/H is connected $\Rightarrow G$ is connected
2. Let G be a locally compact T_2 group and μ be a left Haar measure on G . Then show that the following are equivalent:
(i) G is σ -compact;
(ii) μ is σ -finite;
(iii) collection of mutually disjoint sets of positive measure is countable;
(iv) for every non-empty open set U there exists a sequence (x_n) in G such that $G = \cup Ux_n$

3. a) Let G be a locally compact T_2 group and m be a left Haar measure on G . Then $m(E^{-1}) = \int_E \frac{1}{\Delta(x)} dm(x)$ where $\Delta : G \rightarrow \mathbb{R}^+$ is the modular function.
- b) What are the left & right Haar measures on $GL_n(\mathbb{R})$? Justify your answer?
4. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition of d and V_λ denotes the irreducible representation of S_d corresponding to λ . Then prove that
- (i) $V_{(1, \dots, 1)}$ is the alternating representation of S_d
 - (ii) $V_{(2, 1)}$ is the standard representation of S_3
 - (iii) $V_{\lambda'} \cong V_\lambda \otimes U'$ where λ' is the conjugate partition to λ and U' is the alternating representation.